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Some constraints and solutions of the Kadomtsev–Petviashvili equation

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Abstract. We illustrate the relations among the symmetry invariant group and constraint for a differential equation. Applied to the Kadomtsev–Petviashvili equation, some constraints and solutions are given. In particular, the equation associated with the symmetry σ of the KP equation is introduced and discussed.

1. Symmetry, invariant group and constraint

We consider

$$M = \{u(t, x, y, \ldots)| \in C^{\infty}\}$$

and the differential equation

$$F(t, x, y, \ldots u, u_t, u_x, u_y, \ldots) = 0 \tag{1.1}$$

which is written briefly as

$$F(t, x, y, \ldots u) = 0 \qquad \text{or} \qquad F(u) = 0.$$

Suppose N is a set of the solutions of (1.1), i.e.

$$N = \{u \in M | F(u) = 0\}$$

and $G = \{g\}$ is a Lie group which acts on M:

$$g : M \to M$$
$$u \to \bar{u} = g \circ u \qquad g \in G.$$

Definition 1.1. G is called an invariant group of (1.1), if $g \circ N \subset N$ for any $g \in G$, that is, $\bar{u} = g \circ u$ is a solution of u is a solution of (1.1) [1,2].

In particular, if $G = \{g_\varepsilon | \varepsilon \in R\}$ is a one-parameter invariant group:

$$g_\varepsilon : u \to \bar{u}(u, \varepsilon)$$
$$g_0 \circ u = \bar{u}(u, 0) = u$$

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and $F(\bar{u}) = 0$ is established for any ε if $u \in N$. Considering the Taylor expansion of $F(\bar{u})$ for ε, we have

$$F'(u) \circ \sigma = 0 \tag{1.2}$$

where

$$\sigma = \left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\varepsilon}\right)\bigg|_{\varepsilon=0}$$

and $F'(u) \circ \sigma$ is the derivative of $F(u)$ to the direction σ, i.e.

$$F'(u) \circ \sigma = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(u + \varepsilon\sigma)|_{\varepsilon=0}. \tag{1.3}$$

$F'(u) \circ \sigma$ can also be considered as the action of the direction σ on the function F and written as $\sigma \circ F$.

Definition 1.2. $\sigma(t, x, y, \ldots u, u_t, u_x, u_y, \ldots)$ ($\equiv$ $\sigma(t, x, y, \ldots)$ or $\sigma(u)$) is called a symmetry of differential equation (1.1), if

$$F'(u) \circ \sigma = 0$$

is established for any $u \in N$.

In particular, for the evolution equation

$$u_t = K(t, x, y, \ldots, u, u_x, u_y, \ldots)$$

equation (1.2) is reduced to

$$\sigma_{\mathrm{t}} = K'\sigma \qquad \text{or} \qquad \frac{\partial \sigma}{\partial t} = [k, \sigma]$$

where σ_{t} is the total derivative of σ to t and $[K, \sigma] = K'\sigma - \sigma' K$ [3,4]. Therefore, there is a corresponding symmetry to a one-parameter invariant group of a differential equation. Conversely, there is a corresponding one-parameter invariant group for a symmetry as well.

Theorem 1.1. If $\bar{u} = \bar{u}(u, \varepsilon)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}}{\mathrm{d}\varepsilon} = \sigma(\bar{u}) \\ \bar{u}|_{\varepsilon=0} = u \end{cases} \tag{1.4}$$

where σ is a symmetry of (1.1), then

$$g_\varepsilon : u \to \bar{u}(u, \varepsilon)$$

is a one-parameter invariant group of (1.1) [5].

Definition 1.3. Solution u of the differential equation (1.1) is called group G invariant if u is invariant for the action of any $g \in G$, i.e. $g \circ u = u, g \in G$.

In particular, assume $G = \{g_\varepsilon | \varepsilon \in R\}$ is a one-parameter invariant group of (1.1) corresponding to the symmetry $\sigma(u) = \mathrm{d}\bar{u}/\mathrm{d}\varepsilon|_{\varepsilon=0}$.

Theorem 1.2. If g_ε is a one-parameter invariant group of (1.1) corresponding to the symmetry σ, then solution u is g_ε-invariant if and only if u satisfies $\sigma(u) = 0$ [5].

Therefore, to look for the g_ε-invariant solution, we only need to solve the equations:

$$F(u) = 0 \qquad \sigma(u) = 0. \tag{1.5}$$

It is known [5] that these two equations in (1.5) are compatible and they can be reduced to a lower-dimensional partial differential equation or an ordinary equation. In [5] we discussed the $1 + 1$ dimensional KdV equation. In this paper, we will discuss the $2 + 1$ dimensional KP equation.

2. Constraints of the KP equation

We consider the KP equation

$$u_t + u_{xxx} + 6uu_x - D^{-1}u_{yy} = 0 \tag{2.1}$$

($D^{-1} = \int \mathrm{d}x$) or

$$(u_t + u_{xxx} + 6uu_x)_x - u_{yy} = 0.$$

As is known, the KP equation (2.1) has the following symmetries [6, 4]:

$$K_0 = u_x \qquad K_1 = u_y \qquad K_2 = D^{-1}u_{yy} - u_{xxx} - 6uu_x = u_t$$
$$K_3 = \tfrac{4}{3}D^{-2}u_{yyy} - 4u_{yxx} - 8u_xD^{-1}u_y - 16uu_y, \ldots$$
$$\tau_0 = 3tu_x - \tfrac{1}{2} \qquad \tau_1 = 2tu_y + yu_x \qquad \tau_2 = 3tu_t + 2yu_y + xu_x + 2u, \ldots$$

and the Lax pair [7]:

$$\phi_y = \sqrt{3}\mathrm{i}(\phi_{xx} + u\phi) \tag{2.2}$$

$$\phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi + \sqrt{3}\mathrm{i}(D^{-1}u_y)\phi \tag{2.3}$$

and we have

Lemma 2.1. $\sigma = (\phi\bar{\phi})_x$ is a symmetry of the KP equation (2.1), where $\bar{\phi}$ is the complex conjugate function of ϕ [7, 8].

Proof. By using (2.2) and (2.3), we can check that $\gamma = \phi\bar{\phi}$ satisfies

$$\gamma_0 + \gamma_{xxx} + 6u\gamma_x - D^{-1}\gamma_{yy} = 0$$

and then we have

$$\sigma_t + \sigma_{xxx} + 6u\sigma_x + 6u_x\sigma - D^{-1}\sigma_{yy} = 0.$$

(i) If we take the symmetries K_0, K_1, K_2, K_3, τ_0, τ_1, τ_2, τ_3 or their linear combinations, the KP equation can be constrained to the KdV equation, Boussinesq equation and so on [9]. For example, by using $\sigma = \tau_0 - aK_1 = 3tu_x - 1/2 - au_y$ (a is an arbitrary constant), the KP equation is constrained to the KdV equation

$$f_\tau + f_{\xi\xi\xi} - 6ff_\xi = 0 \tag{2.4}$$

where

$$\xi = x + \frac{3ty}{a} + \frac{3t^3}{a^2} \qquad \tau = t$$

and

$$u = f\left(x + \frac{3ty}{a} + \frac{3t^3}{a^2}, t\right) - \frac{y}{2a}$$

is a solution of the KP equation.

By using the symmetry $\sigma = \tau_1 - aK_2 = 2tu_y + yu_x - au_t$, the KP equation is constrained to the Boussinesq equation

$$f_\xi + f_{\xi\xi\xi} + 6ff_\xi - D^{-1}f_{\eta\eta} = 0 \tag{2.5}$$

where

$$\xi = x + \frac{yt}{a} + \frac{2t^3}{3a^2} \qquad \eta = -\left(y + \frac{t^2}{a}\right)$$

and

$$u = f\left(x + \frac{yt}{a} + \frac{2t^3}{3a^2}, -y - \frac{t^2}{a}\right) - \frac{y}{6a} - \frac{t^2}{6a^2} + \frac{1}{6}$$

is a solution of the KP equation.

(ii) We take the symmetry $\sigma = u_x - (\phi\bar{\phi})_x$. Since

$$u_x - (\phi\bar{\phi})_x = 0$$

we have

$$u = \phi\bar{\phi}.$$

Substituting $u = \phi\bar{\phi}$ into the Lax pair (2.2) and (2.3) of the KP equation (2.1), we can obtain the group-invariant solution corresponding to the symmetry $u_x - (\phi\bar{\phi})_x$ [7], that is, we need to solve the following equations

$$u_x = \phi\bar{\phi}_x + \phi_x\bar{\phi}(u = \phi\bar{\phi}) \tag{2.6}$$

$$\phi_y = \sqrt{3}\mathrm{i}(\phi_{xx} + u\phi) \tag{2.7}$$

$$\phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi + \sqrt{3}\mathrm{i}(D^{-1}u_y)\phi. \tag{2.8}$$

Since (2.6), (2.7) can be reduced to

$$\frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y = -(\phi_x\bar{\phi} - \phi\bar{\phi}_x)$$

and we have

$$\phi_x\bar{\phi} = \frac{1}{2}\left(u_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)$$

$$\bar{\phi}_x\phi = \frac{1}{2}\left(u_x + \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)$$

$$\phi_x\bar{\phi}_x = \frac{1}{4u}\left(u_x^2 + \frac{1}{3}(D^{-1}u_y)^2\right)$$

then (2.8) is reduced to

$$u_t = -4u_{xxx} - 12uu_x + 12(\phi_x\bar{\phi}_x)_x$$

or

$$u_t = -4u_{xxx} - 12uu_x + \left[\frac{3}{u}\left(u_x^2 + \frac{1}{3}(D^{-1}u_y)^2\right)\right]_x . \tag{2.9}$$

Therefore, to look for the group-invariant solutions, we only need to solve the compatible equations (1.6) and (4.9), or the equations (1.6) and

$$3u_{xxx} + 6uu_x + D^{-1}u_{yy} - \left[\frac{3}{u}\left(u_x^2 + \frac{1}{3}(D^{-1}u_y)^2\right)\right]_x = 0. \tag{2.10}$$

Equation (2.10) is a $1+1$ dimensional equation. In the next section, we expand the discussion to the general case and we call (2.10) an associate equation to symmetry u_x of the KP equation.

3. Associate equation to the symmetry σ of the KP equation

In the last section, we obtained a $1+1$ dimensional equation (2.10) which is called an associate equation to the symmetry u_x of the KP equation:

$$3u_{xxx} + 6uu_x + D^{-1}u_{yy} - \left[\frac{3}{u}\left(u_x^2 + \frac{1}{3}(D^{-1}u_y)^2\right)\right]_x = 0. \tag{3.1}$$

Equation (3.1) can be understood as the integrable condition of the following equations:

$$\phi_x = \frac{1}{2u}\left(u_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)\phi \tag{3.2}$$

$$\phi_y = \sqrt{3}\mathrm{i}(\phi_{xx} + u\phi) \tag{3.3}$$

i.e. $\phi_{xy} = \phi_{yx}$ if and only if (3.1) is established. Suppose

$$\psi = \frac{\phi_x}{\phi}$$

then (3.3) is reduced to

$$D^{-1}\psi_y = \sqrt{3}\mathrm{i}(\psi_x + \psi^2 + u)$$

or

$$u = -\psi_x - \psi^2 - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}\psi_y. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain the equation

$$\psi_{xx} - 2\psi^3 - \frac{2\mathrm{i}}{\sqrt{3}}\psi D^{-1}\psi_y - \frac{2\mathrm{i}}{\sqrt{3}}D^{-1}(\psi\psi_y) + \frac{1}{3}D^{-1}\psi_{yy} = 0 \tag{3.5}$$

and call (3.5) the modified equation of (3.1). Since (3.5) is invariant when we change (ψ, y) to $(-\psi, -y)$ and equation (5.1) is invariant when we change y to $-y$, then

$$\bar{u} = \psi_x - \psi^2 - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}\psi_y$$

is a solution of equation (3.1) when u is a solution of equation (3.1). Therefore, we have the Backlund transformation for equation (3.1).

Theorem 3.1. If u is a solution of the equation (3.1), ϕ satisfies (3.2) and (3.3), then

$$\bar{u} = u + 2\left(\frac{\phi_x}{\phi}\right)_x$$

is a solution of the equation (5.1) as well.

Since (3.2)

$$\frac{\phi_x}{\phi} = \frac{1}{2u}\left(u_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)$$

we have:

Corollary 3.1. If u is a solution of the equation (5.1), then

$$\bar{u} = u + \left(\frac{1}{u}\left(u_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)\right)_x$$

is a solution of the equation (5.1) as well.

Example 3.1. $u = -\frac{1}{54}x^2y^{-2}$ is a solution of equation (3.1), according to corollary 3.1, we obtain the solution

$$\bar{u} = -\tfrac{1}{54}x^2y^{-2} - 2x^{-2} + \frac{2\mathrm{i}}{3\sqrt{3}}y^{-1}$$

and then we have the solution

$$\bar{\bar{u}} = \bar{u} + \left(\frac{1}{\bar{u}}\left(\bar{u}_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}u_y\right)\right)_x$$

and so on.

In general, we take the symmetry $\sigma - (\phi\bar{\phi})_x$, where σ is an arbitrary symmetry of the KP equation (2.1), corresponding to the covariant conserved $\gamma(\gamma_x = \sigma)$. Since

$$\phi_x\bar{\phi} + \phi\bar{\phi}_x = \gamma_x$$

and (3.3), we have

$$\phi_x\bar{\phi} - \phi\bar{\phi}_x = -\frac{\mathrm{i}}{\sqrt{3}}D^{-1}\gamma_y$$

and we obtain

$$\phi_x = \frac{1}{2\gamma}\left(\gamma_x - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}\gamma_y\right)\phi \tag{3.6}$$

$$\phi_y = \sqrt{3}\mathrm{i}(\phi_{xx} + u\phi) \tag{3.7}$$

$$\phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi + \frac{\mathrm{i}}{\sqrt{3}}(D^{-1}u_y)\phi. \tag{3.8}$$

To look for the group-invariant solutions corresponding to the symmetry $\sigma - (\phi\bar{\phi})_x$, we need to solve equations (3.6)–(3.8). Since (3.6), (3.7) can be reduced to

$$\phi_y = \sqrt{3}\mathrm{i}\left(-\frac{\gamma_x^2}{4\gamma^2} + \frac{\gamma_{xx}}{2\gamma} - \frac{1}{12\gamma^2}(D^{-1}\gamma_y)^2 + u\right)\phi + \frac{\gamma_y}{2\gamma}\phi$$

then by using $\phi_{xy} = \phi_{yx}$, we obtain the equation

$$\left(-\frac{\gamma_x^2}{4\gamma^2} + \frac{\gamma_{xx}}{2\gamma} - \frac{1}{12\gamma^2}(D^{-1}\gamma_y)^2 + u\right)_x = -\left(\frac{1}{6\gamma}D^{-1}\gamma_y\right)_y$$

i.e.

$$3\gamma_{xxx} + 6\gamma u_x + D^{-1}\gamma_{yy} - \left(3\frac{\gamma_x^2}{\gamma} + \frac{1}{\gamma}(D^{-1}\gamma_y)^2\right)_x = 0. \tag{3.9}$$

This is a $1 + 1$ dimensional equation (we can assume that σ (or γ) does not include u_t since it can be replaced by $D^{-1}u_{yy} - u_{xxx} - 6uu_x$ and t is considered as a parameter). To look for the group-invariant solution, we only need to solve the compatible equations (2.1) and (3.9), and we call equation (3.9) the associate equation to the symmetry σ of the KP equation.

Example 3.2. When $\sigma = u_x$, (3.9) is reduced to equation (3.1).

Example 3.3. When $\sigma = u_y$, $\gamma = D^{-1}u_y$ and (3.9) is reduced to the equation

$$3u_{xxy} + 6u_x D^{-1}u_y + D^{-2}u_{yyy} - \left(3\frac{u_y^2}{D^{-1}u_y} + \frac{1}{D^{-1}u_y}(D^{-2}u_{yy})^2\right)_x = 0 \tag{3.10}$$

and (3.6) is reduced to

$$\phi_x = \frac{1}{2D^{-1}u_y}\left(u_y - \frac{\mathrm{i}}{\sqrt{3}}D^{-2}u_{yy}\right)\phi. \tag{3.11}$$

Substituting

$$u = -\psi_x - \psi^2 - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}\psi_y \left(\psi = \frac{\phi_x}{\phi}\right)$$

into (3.11), we obtain the modified equation of (3.10):

$$\psi_{xy} - 4\psi D^{-1}(\psi\psi_y) + \frac{1}{3}D^{-3}\psi_{yyy} - \frac{2\mathrm{i}}{\sqrt{3}}\psi D^{-2}\psi_{yy} - \frac{2\mathrm{i}}{\sqrt{3}}D^{-2}(\psi\psi_y)_y = 0. \tag{3.12}$$

Since (3.12) is invariant when we change (ψ, y) to $(-\psi, -y)$, and (3.10) is invariant when we change y to $-y$, we obtain the Backlund transformation for the equation (3.10).

Theorem 3.2. If u is a solution of the equation (3.10), ϕ satisfies equations (3.11) and (3.13), then

$$\bar{u} = u + 2\left(\frac{\phi_x}{\phi}\right)_x$$

is a solution of (3.10) as well.

Corollary 3.2. If u is a solution of the equation (3.10) then

$$\bar{u} = u + \left(\frac{1}{D^{-1}u_y}\left(u_y - \frac{\mathrm{i}}{\sqrt{3}}D^{-2}u_{yy}\right)\right)_x$$

is a solution of (3.10) as well.

Example 3.4. We take $\sigma = 3tu_x - \frac{1}{2}(\gamma = 3tu - \frac{1}{2})$; (3.9) and (3.5) are reduced to

$$3u_{xxx} + 6uu_x + D^{-1}u_{yy} - \frac{xu_x}{t} - \left(\frac{(3tu_x - 1/2)^2}{t(3tu - x/2)} + \frac{3t}{3tu - x/2}(D^{-1}u_y)^2\right)_x = 0 \quad (3.13)$$

and

$$\phi_x = \frac{1}{2(3tu - x/2)}(3tu_x - \tfrac{1}{2} - \mathrm{i}\sqrt{3}tD^{-1}u_y)\phi. \quad (3.14)$$

Substituting

$$u = -\psi_x - \psi^2 - \frac{\mathrm{i}}{\sqrt{3}}D^{-1}\psi_y$$

into (3.14), we have

$$\psi_{xx} - 2\psi^3 + \frac{1}{3}D^{-1}\psi_{yy} - \frac{2\mathrm{i}}{\sqrt{3}}\psi D^{-1}\psi_y - \frac{2\mathrm{i}}{\sqrt{3}}D^{-1}(\psi\psi_y) - \frac{x\psi}{3t} + \frac{1}{6t} = 0. \quad (3.15)$$

When we change (ψ, y) to $(-\psi, -y)$, equation (3.5) is reduced to

$$\psi_{xx} - 2\psi^3 + \frac{1}{3}D^{-1}\psi_{yy} - \frac{2\mathrm{i}}{\sqrt{3}}\psi D^{-1}\psi_y - \frac{2\mathrm{i}}{\sqrt{3}}D^{-1}(\psi\psi_y) - \frac{x\psi}{3t} - \frac{1}{6t} = 0. \quad (3.16)$$

Therefore, we could not obtain the auto-Backlund transformation for the equation (3.13). In this case, (3.16) is equivalent to

$$\phi_x = \frac{1}{2(3tu - x/2)}(3tu_x + \tfrac{1}{2} - \mathrm{i}\sqrt{3}tD^{-1}u_y)\phi \quad (3.17)$$

$$\phi_y = \sqrt{3}\mathrm{i}(\phi_{xx} + u\phi) \quad (3.18)$$

and the integrable condition of (3.18) and (3.19) is

$$3u_{xxx} + 6uu_x + D^{-1}u_{yy} - \frac{xu_x}{t} - \left(\frac{(3tu_x - 1/2)^2}{t(3tu - x/2)} + \frac{3t}{3tu - x/2}(D^{-1}u_y)^2\right)_x + \mathrm{i}\frac{3tu_y}{\sqrt{3}(3tu - x/2)} = 0. \quad (3.19)$$

Therefore, we obtain:

Theorem 3.3. If u is a solution of the equation (3.13), then

$$\bar{u} = u + 2\left(\frac{\phi_x}{\phi}\right)_x$$

or

$$\bar{u} = u + \left(\frac{1}{(3tu - x/2)}(3tu_x - \tfrac{1}{2} - \mathrm{i}\sqrt{3}tD^{-1}u_y)\right)_x$$

is a solution of (3.19), where ϕ satisfies (3.3) and (3.14).

These examples show the difference between the K-symmetries (examples 3.2 and 3.3) and the τ-symmetries (example 3.4). We can compare with the conclusions in [12–14], and extend the discussion to the general cases.

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